DERIVATIVES

The derivative of the function $f(x)$ in the point x_0 is the limit to which the ratio of the increment of the function $f(x+\Delta x)-f(x)$ tends to the increment of the variable Δx, when Δx tends to 0:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The derivative carries information about how the function changes at a given point.

We denote the derivative of the function $f(x)$:

$$f'(x), \frac{df(x)}{dx}$$

or

$$y', \frac{dy}{dx}, \dot{y}$$

Finding the derivative of a function is called function differentiation.

The derivative of a constant function is zero:

funkcji stałej równa się zeru:

a - constant

$$a' = 0$$

Derivative x^a:

$$(x^a)' = ax^{a-1}$$

The derivative of the product of a constant by a function is equal to the product of the constant and its derivative

$$\left(k \cdot f(x)\right)' = k \cdot f'(x)$$

Derivative of sum / difference of a function:

$$\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$$

The derivative of the product of a function:

$$\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

The derivative of the quotient of a function:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

Derivative of a complex function:

$$\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$$

$$(\log_a x)' = \frac{1}{xlna}$$

$$(lnx)' = \frac{1}{x}$$

$$(a^x)' = a^x lna$$

$$(e^x)' = e^x$$

Examples:

$$(-3)' = 0$$

$$(x^4)' = 4x^3$$

$$\left(\sqrt{x}\right)' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\left(\frac{1}{x}\right)' = (x^{-1})' = -x^{-2} = -\frac{1}{x^2}$$

$$\left(4x^2\sqrt{x}\right)' = \left(4x^{\frac{5}{2}}\right)' = 10x^{\frac{3}{2}}$$

$$(2^x)' = 2^x ln2$$

$$(\log_3 x)' = \frac{1}{xln3}$$

$$\left(x^2 + \sqrt[4]{x}\right)' = 2x + \frac{1}{4}x^{-\frac{3}{4}}$$

$$(x^6e^x)' = (x^6)'e^x + x^6(e^x)' = 6x^5e^x + x^6e^x$$

$$\left(\frac{lnx}{x^2}\right)' = \frac{(lnx)'x^2 - lnx(x^2)'}{x^4} = \frac{\frac{1}{x}x^2 - lnx \cdot 2x}{x^4} = \frac{x(1 - 2lnx)}{x^4}$$

$$\left(\sqrt{4x^2 + 3x}\right)' = \frac{8x + 3}{2\sqrt{4x^2 + 3x}}$$

- Study of the course of function variability
-
- • If the derivative of a function is positive in a certain interval, then the function is increasing in this interval.
- • If the derivative of a function is negative in a certain interval, then the function decreases in this interval.
- • If the derivative of a function is equal to zero at every point in a certain interval, then the function is constant in that interval
- • If a function has a local extremum at some point, then the derivative of the function at that point equals zero.
- • If the derivative of the function changes the sign from negative to positive while the variable x passes through the point x_0, then the function reaches its minimum.
- • If the derivative of the function changes the sign from positive to negative when the variable x passes through the point x_0, then the function reaches its maximum.

Example 2. Examine the monotonicity of a function:

$$f(x) = x^3 + 3x^2 - 9x - 2$$

- • The domain of the function is the set of real numbers.
- • The function has no asymptotes.

$\lim_{x \to -\infty} x^3 + 3x^2 - 9x - 2 = -\infty$

$\lim_{x \to +\infty} x^3 + 3x^2 - 9x - 2 = \infty$

- We calculate the derivative:

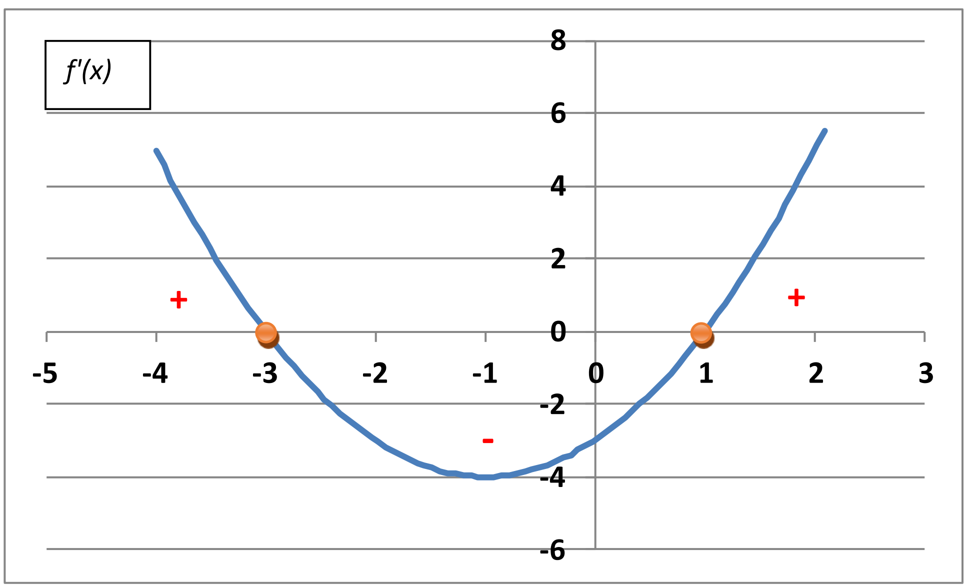
$f'(x) = 3x^2 + 6x - 9$

We equate the derivative to zero in order to find the local extremes of the function:

$$3x^2 + 6x - 9 = 0$$

$$x^2 + 2x - 3 = 0$$

$$x = -3 \text{ and } x = 1$$



Local extremes are for the value of a variable $x = -3$ and $x = 1$.

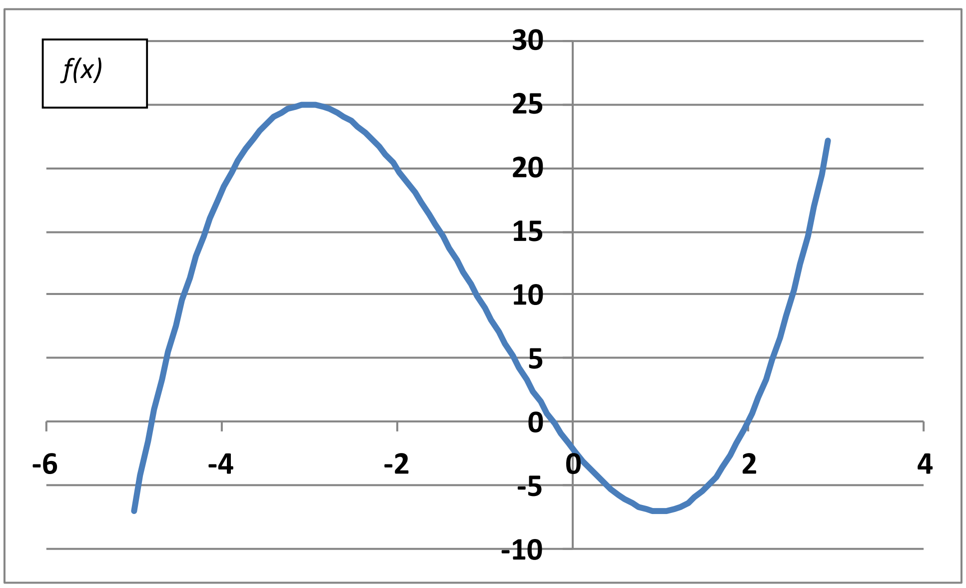
In the interval $x \in (-\infty, -3)$ and $x \in (1, \infty)$ the derivative of the function $f(x)$ is positive, i.e. the function $f(x)$ increases in these intervals.

In the interval $x \in (-3,1)$ the derivative of the function $f(x)$ is negative, i.e. the function $f(x)$ is decreasing in this interval.

We can create a table of the function variability course:

$\boldsymbol{x}$	$-\infty$	$\cdots$	**-3**	$\cdots$	**1**	$\cdots$	∞
$\boldsymbol{f'(x)}$	∞	$+$	**0**	$-$	**0**	$+$	$+$
$\boldsymbol{f(x)}$	$-\infty$	↗	**25**	↘	**-7**	↗	∞

And the graph of the function $f(x) = x^3 + 3x^2 - 9x - 2$:



Example:

$$f(x) = \frac{2}{1 + 3e^{-x}}$$