Sequences

If each integer n is assigned one real number a_n, we say that an infinite sequence of numbers has been defined. The sequence is written in the form: $a_1, a_2, \dots, a_n$ lub $\{a_n\}$

The numbers $a_1, a_2, \dots$ are called the terms of the sequence $\{a_n\}$, and a_n is the general term of the sequence.

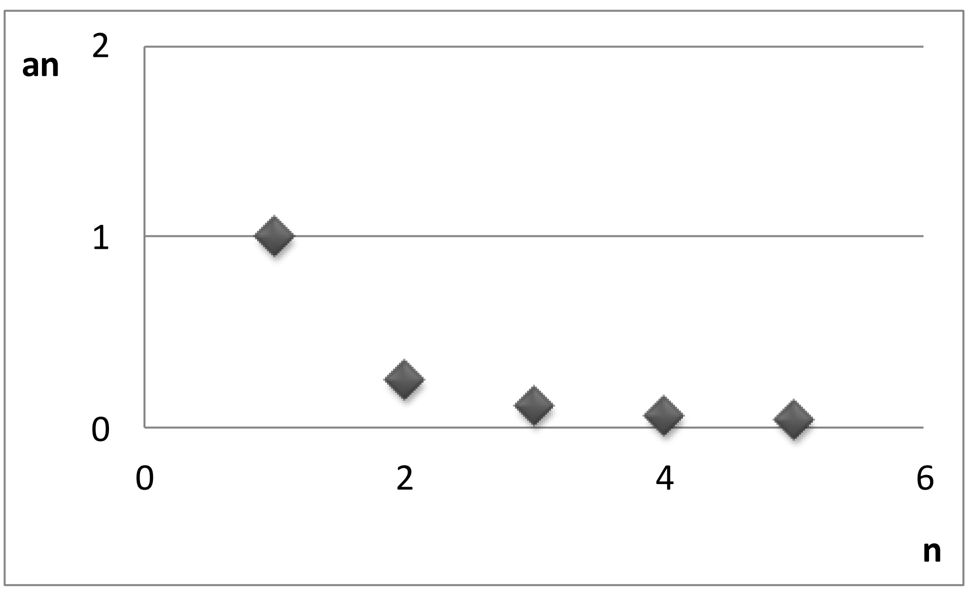
The sequence $\{a_n\}$ has a limit g, when with $n \to \infty$, $a_n \to g$ or $\lim\limits_{n\to\infty} a_n = g$

Example 1: $a_n = \frac{1}{n^2}$

n	1	2	3	4	5	…	$\to \infty$
a_n	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$	…	$\to 0$

The sequence $\{a_n\}$ has a limit 0, when $n \to \infty$,

$\frac{1}{n^2} \to 0$, when $n \to \infty$, or $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$



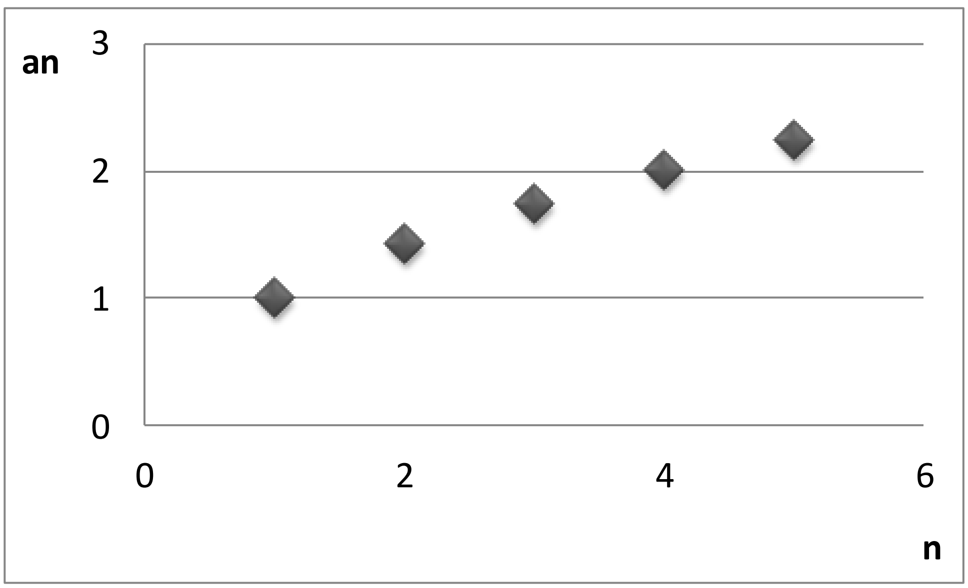
Example 2:

$$a_n = \sqrt{n}$$

n	1	2	3	4	5	…	$\to \infty$
a_n	1	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{4}$	$\sqrt{5}$	…	$\to \infty$

The sequence $\{a_n\}$ has a limit ∞, when $n \to \infty$,

$\sqrt{n} \to \infty$, when $n \to \infty$, or $\lim_{n\to\infty} \sqrt{n} = \infty$



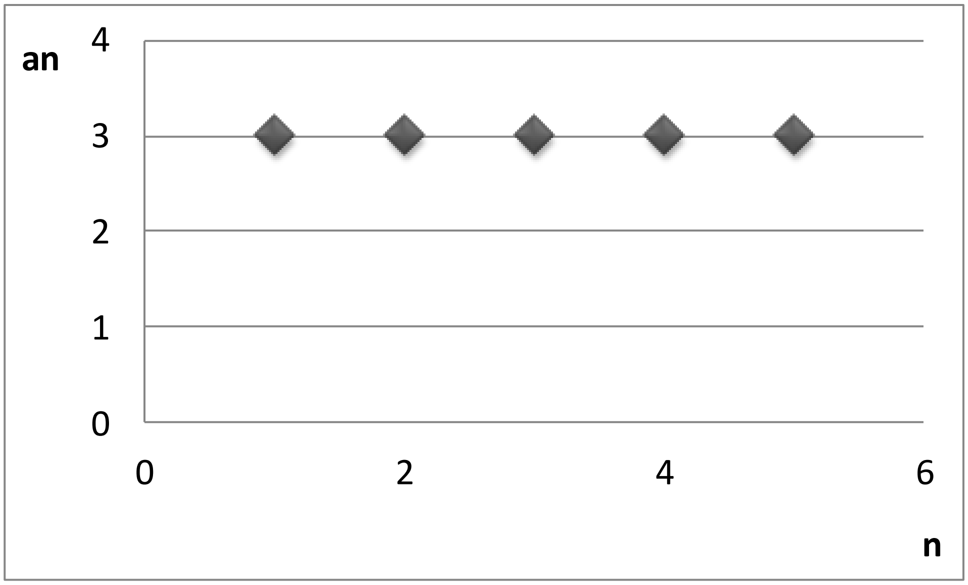
Example 3:

$$a_n = 3$$

n	1	2	3	4	5	…	$\to \infty$
a_n	3	3	3	3	3	…	$\to 3$

The sequence $\{a_n\}$ has a limit 3, when $n \to \infty$,

$3 \to 3$, when $n \to \infty$, or $\lim_{n\to\infty} 3 = 3$

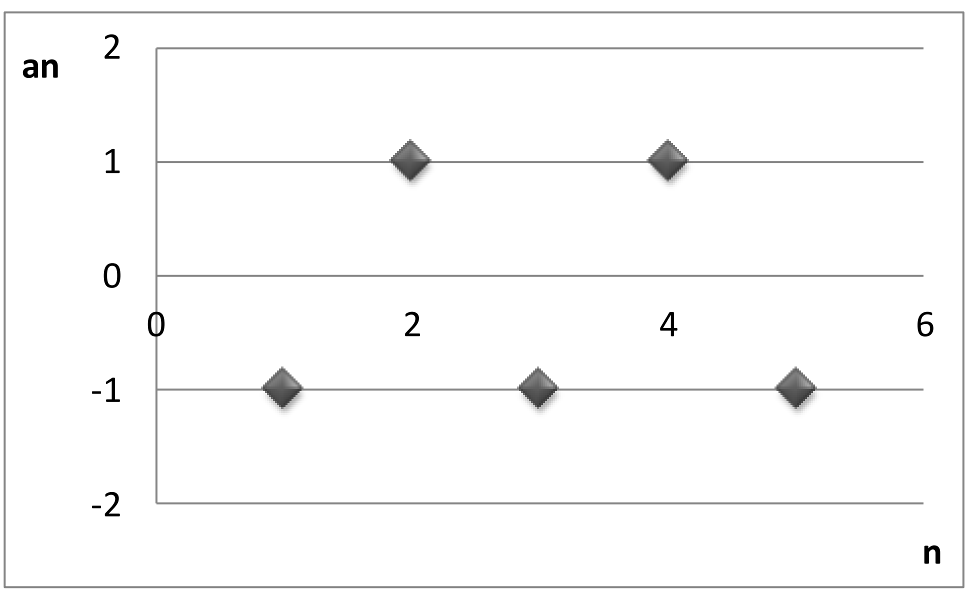


Example 4:

$$a_n = (-1)^n$$

n	1	2	3	4	5	…	$\to \infty$
a_n	-1	1	-1	1	-1	…	$\to ?$

The sequence $\{a_n\}$ it is not known what the limit is, when $n \to \infty$



DEFINITIONS:

Definition 1. A sequence is recursively specified when the value of the first element (or the first few elements) is given, from which the next element in the sequence is determined using the formulated formula.

Arithmetic sequence: given are numbers a i r

$$\begin{cases} a_1 = a \\ a_{n+1} = a_n + r \end{cases}$$

Geometric sequence: numbers are given a i q

$$\begin{cases} a_1 = a \\ a_{n+1} = a_n \cdot q \end{cases}$$

Fibonacci Sequence: from the third element of the sequence, each element is the sum of the previous two elements

$$\begin{cases} a_1 = 0 \\ a_2 = 1 \\ a_{n+2} = a_{n+1} + a_n \end{cases}$$

Example: Calculate the first five elements of the sequence:

$$\begin{cases} a_1 = 1 \\ a_{n+1} = 5a_n - 3 \end{cases}$$

Solution:

n	1	2	3	4	5
a_n	1	2	7	32	157

Example: Calculate the first five elements of the sequence:

$$\begin{cases} a_1 = 1 \\ a_2 = 2 \\ a_{n+2} = \dfrac{a_{n+1}}{a_n} \end{cases}$$

n	1	2	3	4	5
a_n	1	2	2	1	$\frac{1}{2}$

Example: Determine recursively the arithmetic sequence (a_n), based on:

$$a_1 = -1, a_7 = -13$$

Solution: $a_2 = -1 + r$, $a_3 = -1 + r + r, a_4 = -1 + r + r + r, \ldots,\ a_n = -1 + (n-1) \cdot r$

$$a_7 = -1 + 6 \cdot r = -13 \ \Rightarrow (7-1) \cdot r = -12 \ \Rightarrow r = -2$$

$$\begin{cases} a_1 = -1 \\ a_{n+1} = a_n - 2 \end{cases}$$

Example: Determine recursively the geometric sequence (a_n), based on:

$$a_2 \cdot a_3 = 432, \qquad a_4 = 108$$

$a_2 = a_1 \cdot r$, $a_3 = a_1 \cdot r \cdot r,\ a_4 = a_1 \cdot r \cdot r \cdot r, \ldots,\ a_n = a_1 \cdot r^{n-1}$

$$a_2 \cdot a_3 = a_1 \cdot r \cdot a_1 \cdot r \cdot r = {a_1}^2 \cdot r^3 = 432$$

$$a_4 = a_1 \cdot r^3 = 108$$

$$a_1 \cdot r = \frac{108}{r^2}$$

$${a_1}^2 \cdot r^2 = \frac{432}{r} = \frac{11664}{r^4} \Rightarrow r = 3\ i\ a = 4$$

$$\begin{cases} a_1 = 4 \\ a_{n+1} = a_n \cdot 3 \end{cases}$$

Definition 2. The infinite sequence (a_n) is called increasing if $a_{n+1} > a_n$ for every n. The infinite sequence (a_n) is called decreasing if $a_{n+1} < a_n$ for every n.

The infinite sequence (a_n) is called non-decreasing if $a_{n+1} \geq a_n$ for every n. The infinite sequence (a_n) is called non-increasing if $a_{n+1} \leq a_n$ for every n.

Increasing, non-decreasing, decreasing, and non-increasing sequences are called monotonic sequences.

Example: Prove that $a_n = \frac{n}{n+1}$ is an increasing sequence.

For every n there must be $a_{n+1} - a_n > 0$.

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0$$

Example: Prove that $a_n = \frac{1}{n}$ is an decreasing sequence.

For every n there must be $a_{n+1} - a_n > 0$.

$$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{n - n - 1}{(n+1)n} = \frac{-1}{(n+1)n} < 0$$

An infinite sequence with a finite limit is called a convergent sequence. We call other sequence divergent. If the sequence goes to $+\infty$ we call it the sequence divergent to plus infinity. If the sequence goes to $-\infty$ we call it the sequence divergent to minus infinity.

Definition 3. The number g is called the limit of the sequence (a_n), if for each positive number ε there exists such a natural number n_0, that for $n > n_0$ the inequality $|a_n - g| < \varepsilon$ is met. The sequence which has the limit g is said to converge to the number g. Then we write $\lim_{n\to\infty} a_n = g$.

In other words, when choosing an arbitrarily small distance ε, the sequence terms with sufficiently large numbers are less than ε away from g. It is not the initial words that decide whether the string converges, but those with large numbers.

Example: sequence $\left(\frac{1}{n}\right)$ it converges to 0.

In other words, when choosing an arbitrarily small distance ε, the sequence terms with sufficiently large numbers are less than ε away from g. It is not the initial words that decide whether the string converges, but those with large numbers.

Example: The sequence (1 / n) converges to 0.

Dla każdej liczby $\varepsilon > 0$ należy wskazać takie n_0, że dla $n > n_0$ jest spełniona nierówność $\left|\frac{1}{n} - 0\right| < \varepsilon$.

$\frac{1}{n} < \varepsilon,\ n > \frac{1}{\varepsilon}$.

- If the sequence ciąg $\{a_n\}$ has the limit *a* and the sequence $\{b_n\}$ has the limit *b*, to ciąg $\{a_n + b_n\}$ ma granicę *a+b*.

- If the sequence $\{a_n\}$ has the limit *a* and the sequence $\{b_n\}$ has the limit *b*, to ciąg $\{a_n - b_n\}$ ma granicę *a-b*.

- If the sequence $\{a_n\}$ has the limit *a* and the sequence $\{b_n\}$ has the limit *b*, to ciąg $\{a_n \cdot b_n\}$ ma granicę *a·b*.

- If the sequence $\{a_n\}$ has the limit *a* and the sequence $\{b_n\}$ ma granicę the sequence *b* and none of the elements of the sequence $\{b_n\}$ equals zero or its limit is not zero, then the sequence of quotients is the sequence $\{a_n/b_n\}$ has the limit *a*/*b*.

- If the numerator and denominator of the fraction are polynomials of the same degree with respect to the natural variable n, then the limit of such fraction at $n \to \infty$ r is equal to the quotient of the coefficients at the highest powers of n.

- If the denominator of a fraction is a polynomial of greater degree in relation to the natural variable n than the numerator, then the limit of such fraction at $n \to \infty$ is equal 0.

- If the numerator of the fraction is a polynomial of degree higher in relation to the natural variable n than the denominator, then the absolute value of such fraction with $n \to \infty$ it tends to plus infinity.

- The sequence $a_n = q^n$ has a limit 0 for $-1 < q < 1$ and 1 for $q = 1$.

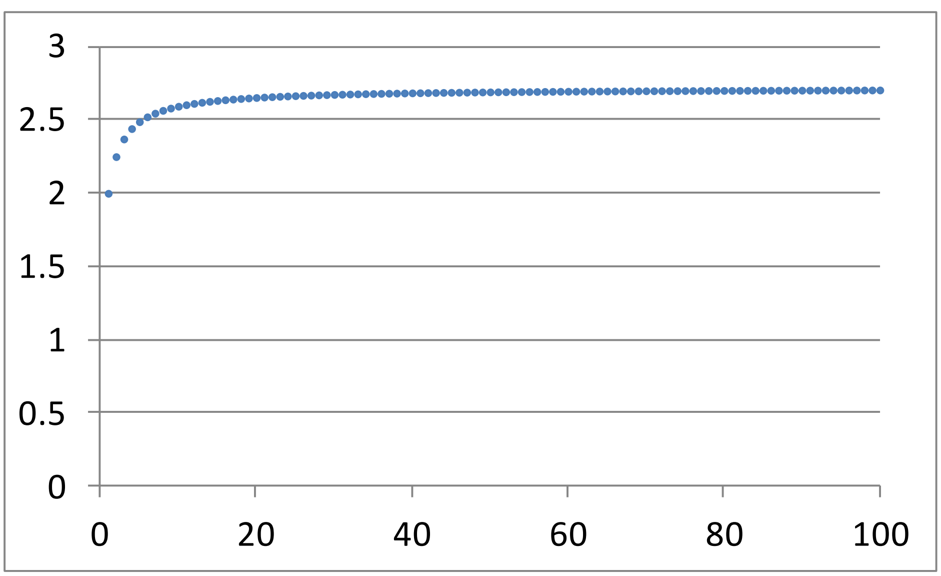
- Three sequence theorem. If the general terms of three sequences $\{a_n\}$ $\{b_n\}$ $\{c_n\}$ are $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = g$, then $\lim_{n\to\infty} b_n = g$.

- $\lim_{n\to\infty} \sqrt[n]{a} = 1$, when $a > 0$

- $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

In general $\lim_{n\to\infty} (1 + a_n)^{\frac{1}{a_n}}$

e numer – Euler number *e* =2,718281828459...



Example 5:

$$a_n = \left(1+\frac{4}{n}\right)^n = \left[\left(1+\frac{4}{n}\right)^{\frac{n}{4}}\right]^4$$

The sequence $\{a_n\}$ has limit e^4, when $n \to \infty$

The number e is the basis of natural logarithms. The exponential function with base e is the inverse of the natural logarithm:

$$lne^x = x$$

$$e^{lnx} = x$$

WolframAlpha

$$a_n = \sqrt{4n^2+5n-7} - 2n$$

```
limit(4*n^2+5*n-7)^0.5-2*n as n->infinity

press „="
```

```
limit (1+1/n)^n as n->infinity

press „="
```

plot for n from 1 to 50:

```
discreteplot (4*n^2+5*n-7)^0.5-2*n, {n,1,50}

press „="
```

When counting the limit of a sequence or a function, the following indefinite expressions can be obtained:

$$\left[\frac{0}{0}\right], \quad \left[\frac{\infty}{\infty}\right], \quad [0 \cdot \infty], \quad [\infty^0], \quad [1^\infty], \quad [0^0]$$

Example:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x} = [1^\infty] = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x} = e^2$$
$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = [1^\infty] = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = e^3$$

Instead, the following expressions (a any positive real $a\epsilon R_+$):

$$[+\infty + a] = \infty$$

$$[-\infty + a] = -\infty$$

$$[+\infty - a] = \infty$$

$$[-\infty - a] = -\infty$$

$$[+\infty \cdot a] = \infty$$

$$[-\infty \cdot a] = -\infty$$

$$[+\infty \cdot (-a)] = -\infty$$

$$[-\infty \cdot (-a)] = +\infty$$

$$[+\infty \cdot (+\infty)] = \infty$$

$$[-\infty \cdot (-\infty)] = \infty$$

$$[+\infty \cdot (-\infty)] = -\infty$$

$$[-\infty \cdot (+\infty)] = -\infty$$

$$\left[\frac{a}{+\infty}\right] = 0^+$$

$$\left[\frac{a}{-\infty}\right] = 0^-$$

$$\left[\frac{-a}{+\infty}\right] = 0^-$$

$$\left[\frac{-a}{-\infty}\right] = 0^+$$

$$\left[\frac{a}{0^+}\right] = +\infty$$

$$\left[\frac{a}{0^-}\right] = -\infty$$

$$\left[\frac{-a}{0^+}\right] = -\infty$$

$$\left[\frac{-a}{0^-}\right] = +\infty$$

$$[\infty^a] = \infty$$

$$[\infty^\infty] = \infty$$

where the symbol $[0^+]$ s the limit of a positive function that is zero, and the symbol $[0^-]$ is the limit of a negative function that is zero.

FUNCTIONS

If every number x from one set is assigned one number y from the other set, then we say that some function has been determined:

$$y = f(x)$$

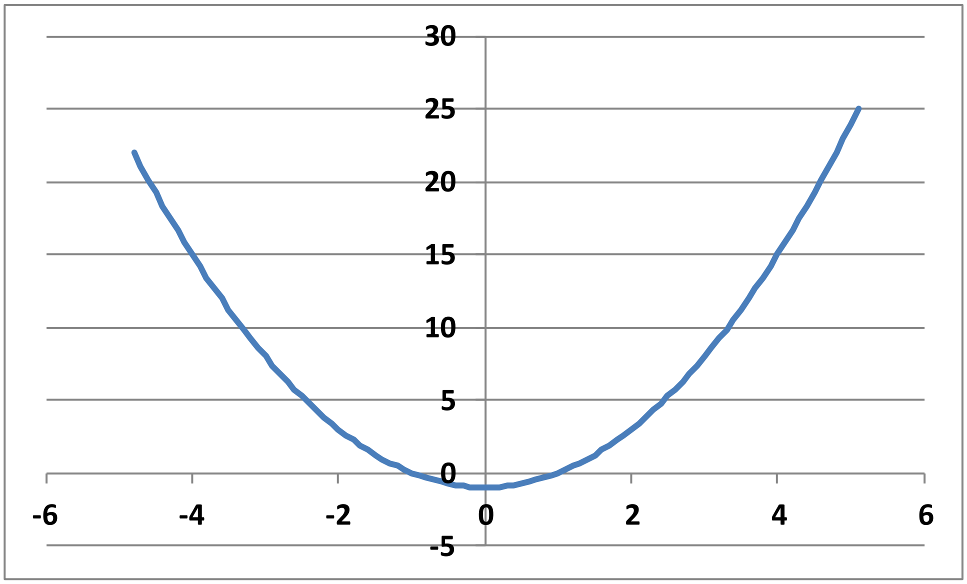
We call the number x the argument of the function, and the number y the value of the function y.

The set of arguments to a function is called the domain of the function.

Example:

$$f(x) = x^2 - 1$$

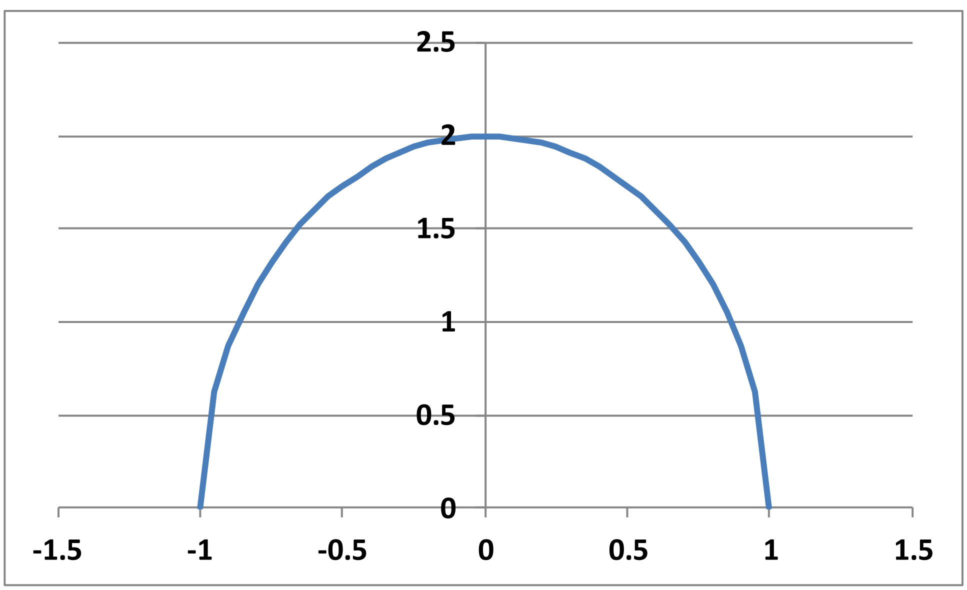
The domain of the function is the set of all real numbers, i.e. $x \in R$ or $-\infty < x < \infty$.



Example:

$$f(x) = 2\sqrt{1 - x^2}$$

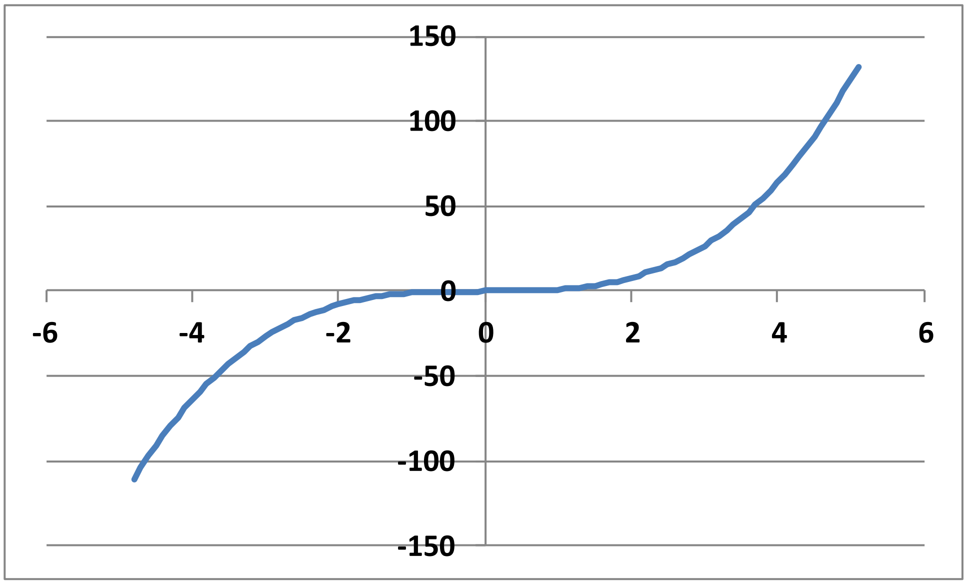
The domain of the function is the set of real numbers which $-1 \leq x \leq 1$.



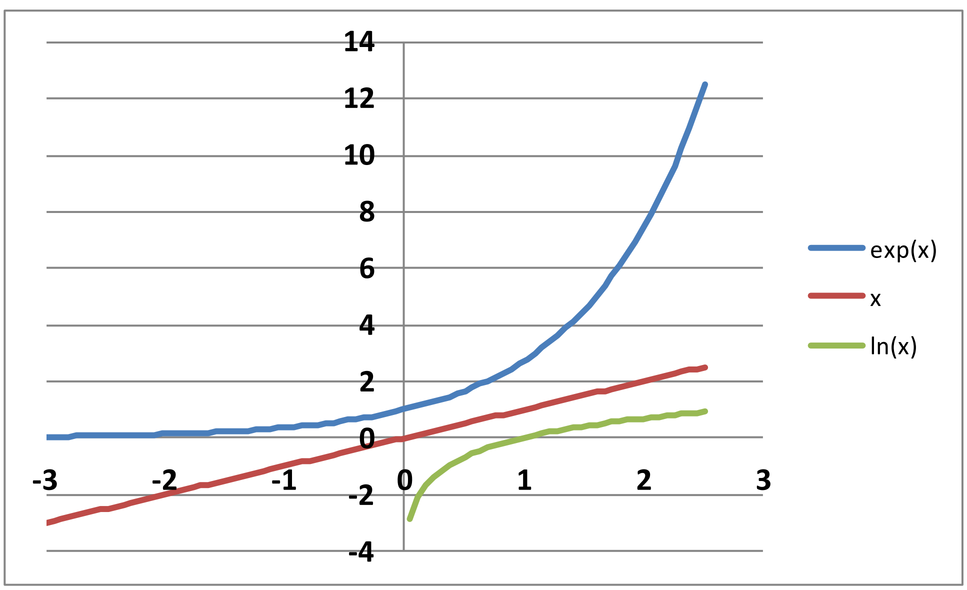
Example:

$$f(x) = x^3$$

The domain of the function is the set of all real numbers, i.e. $x \in R$ or $-\infty < x < \infty$.



Example: Inverse function



$$exp(x) = e^x$$

$$\ln(x) = lnx = \log_e x$$

FUNCTION LIMITS

Left-hand limit of the function:

$$\lim_{x \to {x_0}^-} f(x)$$

Right-hand limit of the function:

$$\lim_{x \to {x_0}^+} f(x)$$

Function $f(x)$ we call it continuous at the point $x = x_0$, if there is a limit $\lim\limits_{x \to x_0} f(x)$ and it is equal to$f(x_0)$.

If $\lim\limits_{x \to {x_0}^-} f(x) = \pm\infty$ or $\lim\limits_{x \to {x_0}^+} f(x) = \pm\infty$, then we say that the function $f(x)$ has a vertical asymptote at point x_0.

If $\lim\limits_{x \to \pm\infty} f(x) = a$, then we say that the line $y = a$ is a horizontal asymptote of the function $f(x)$.

Example: Determine the limits of the function $f(x)$ with $x \to +\infty$, $x \to -\infty$,

and at the points $x = 2$ and $x = -2$.

$$f(x) = \frac{x-1}{x^2-4}$$

$$\lim_{x \to -\infty} \frac{x-1}{x^2-4} = 0$$

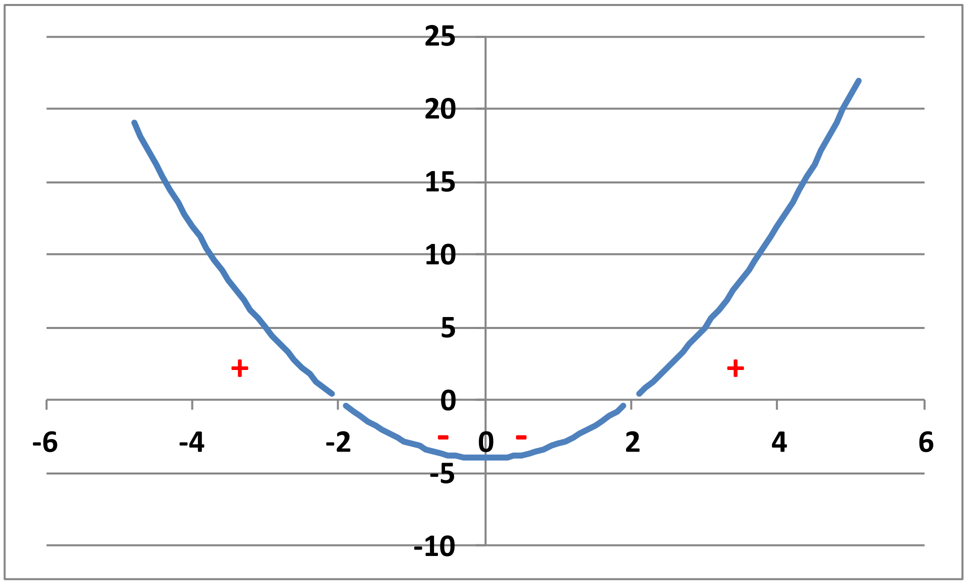
$$\lim_{x \to +\infty} \frac{x-1}{x^2-4} = 0$$

$$\Longrightarrow$$

$\boxed{\text{There is a horizontal asymptote } y = 0.}$

$$f(x) = \frac{x-1}{x^2-4} = \frac{x-1}{(x-2)(x+2)}$$

Denominator plot $f(x) = (x-2)(x+2)$:



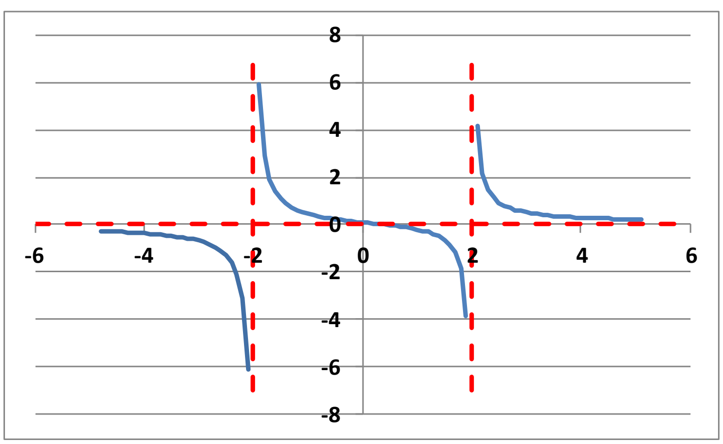
$$\lim_{x\to -2^-} \frac{x-1}{(x-2)(x+2)} = \frac{-3}{0^+} = -\infty$$

$$\lim_{x\to -2^+} \frac{x-1}{(x-2)(x+2)} = \frac{-3}{0^-} = +\infty$$

There are vertical asymptotes $x = -2$ i $x = 2$.

$$\lim_{x\to 2^-} \frac{x-1}{(x-2)(x+2)} = \frac{1}{0^-} = -\infty$$

$$\lim_{x\to 2^+} \frac{x-1}{(x-2)(x+2)} = \frac{1}{0^+} = +\infty$$



Example:

$$f(x) = \frac{1}{e^x + 1}$$

The domain of the function is the set of real numbers.

The function has no zeros. There is no such value x_0, where $f(x_0) = 0$.

$$f(x = 0) = \frac{1}{2}$$

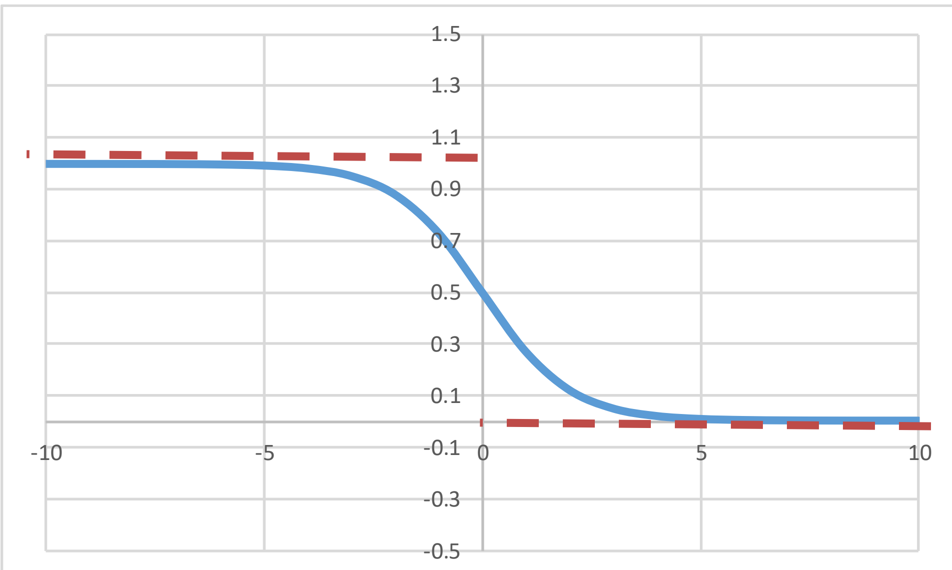
We check the existence of horizontal asymptotes

:

There is a horizontal asymptote
left-hand $y = 1$
and
right-hand $y = 0$.

$\lim\limits_{x \to +\infty} \frac{1}{e^x + 1} = 0$

$\lim\limits_{x \to -\infty} \frac{1}{e^x + 1} = 1$



The number g s called the limit of the function$f$ at the point x_0, if for each sequence (x_n) of arguments of the function f belonging to the interval (a, b), but different from x_0, converging to x_0, the sequence $(f(x_n))$ of the function value converges to the limit$g$. Then we write $\lim\limits_{x \to x_0} f(x) = g$.

The function f is continuous at x_0, if the following conditions are met: x_0 belongs to the function domain f, there is a finite limit of a function as x approaches x_0, $f(x_0) = \lim\limits_{x \to x_0} f(x)$.

Example. The function $y = x^2$ is continuous at the point x_0=2, x_0 belongs to the function domain f, $f(2) = 4$ here is a finite limit of a function, as x approaches x_0, $\lim\limits_{x \to 2} x^2 = 4$

$f(2) = \lim\limits_{x \to 2} x^2$