DIFFERENTIAL EQUATIONS

A first-order differential equation is an equation

$$F(x, y, y') = 0$$

In a differential equation y' is necessary.

The general solution to a differential equation is a function

$$y = f(x, C)$$

If the value of parameter C is known, the function is a special solution to the differential equation

$$y = f(x)$$

If the function being searched for depends only on one independent variable, we say the equation is ordinary.

Example 1.

$$y' = 2y$$

$$\frac{dy}{dx} = 2y$$

$$\frac{1}{y}dy = 2dx$$

$$\int \frac{1}{y}dy = \int 2dx$$

$$lny = 2x + C$$

General solution: $y = e^{2x+C}$

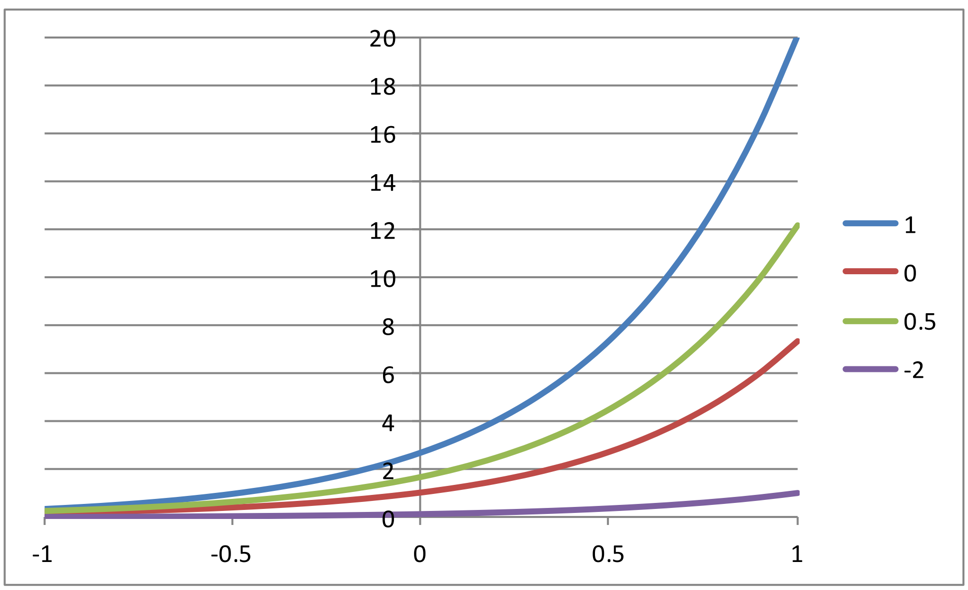
depending on the parameter C we can have for example C:

$y = e^{2x+1}$ dla $C = 1$

$y = e^{2x}$ dla $C = 0$

$y = e^{2x+\frac{1}{2}}$ dla $C = \frac{1}{2}$

$y = e^{2x-2}$ dla $C = -2$



A detailed solution is found after taking into account the initial conditions that will allow to isolate a curve passing through a point (x_0, y_0).

Suppose the starting conditions of Example 1 are as follows $y(0) = 4$:

$$4 = e^{2\cdot 0 + C} = e^{2\cdot 0} e^{C} = 1 \cdot e^{C}$$

$$4 = e^{C}$$

special solution: $y = 4e^{2x}$

Example 2.

$$\sqrt{x} y' = y + 1$$

initial condition

$$y(0) = 1$$

Example 3.

In order to formulate the law of radioactive decay, let us assume that the probability of decay per time unit for a single nucleus is constant and is equal to λ. The number of dN of nuclei that decay in time dt is proportional to the number of N of radioactive nuclei:

$$\frac{dN}{dt} = -\lambda N$$

The initial conditions:

$$N(t = 0) = N_0$$

$$N\left(t = T_{\frac{1}{2}}\right) = \frac{N_0}{2}$$

Application of differential equations to modeling the population in continuous time.

The Malthus model

Model assumptions:

- the population has very good conditions for development, unlimited access to food and breeding sites;

- we only observe the process of reproduction;

- individuals in a given population are the same and are subject to the same laws;

- individuals in a given population are evenly distributed in space;

- the individual is born fully formed, capable of reproduction and can reproduce at any age;

- the moments of reproduction are uniformly distributed in any period of time;

- each individual gives birth to offspring every τ time units, τ is fixed and the same for all individuals;

- each parent has λ descendant individuals.

The mean size of the population $N(t)$ at time t.

Growth in numbers in a short time Δt:

$$\lim_{\Delta t \to 0} \frac{N(t + \Delta t) - N(t)}{\Delta t}$$

is proportional to λ of descendants in the number of τ time units and to the size of the population at a given moment:

$$\lim_{\Delta t \to 0} \frac{N(t + \Delta t) - N(t)}{\Delta t} = \frac{\lambda}{\tau} N(t)$$

$$\frac{dN}{dt} = \frac{\lambda}{\tau} N(t)$$

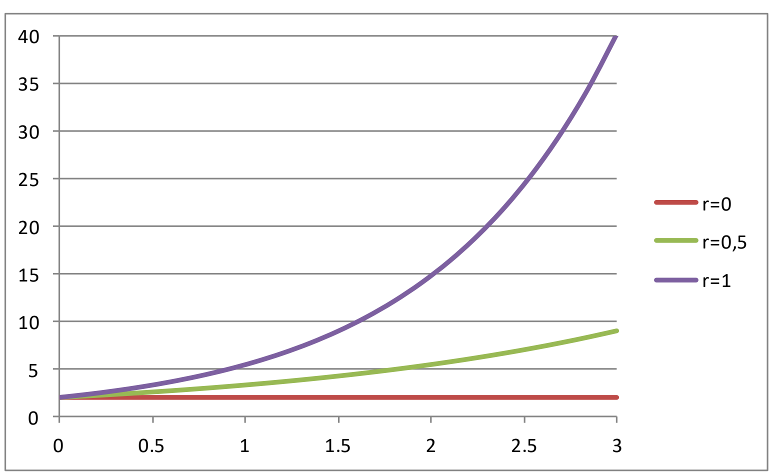
$r = \frac{\lambda}{\tau}$ population fertility rate

$$\frac{dN}{dt} = rN(t)$$

We assume, that $N(t) = N_0$

solution

$$N(t) = N_0 e^{rt}$$



$$\text{jeśli } r > 0 \quad \lim_{t\to\infty} N_0 e^{rt} = +\infty$$

Processes of reproduction and mortality

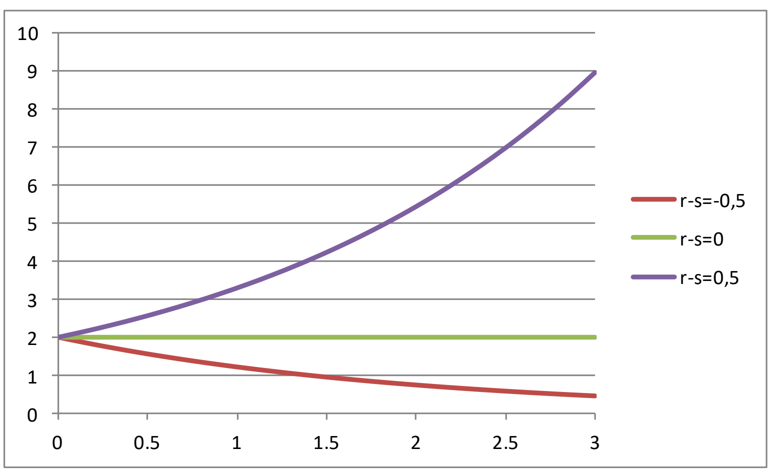
The model should also take into account the mortality of the population:

$$\frac{dN}{dt} = rN(t) - sN(t)$$

$$\frac{dN}{dt} = (r - s)N(t)$$

s - population mortality rate

$$N(t) = N_0 e^{(r-s)t}$$



Verhulst model

$$\frac{dN}{dt} = r(1 - \frac{N}{K})N$$

K- constant characterizing food resources jeżeli $N > K$ populacja maleje

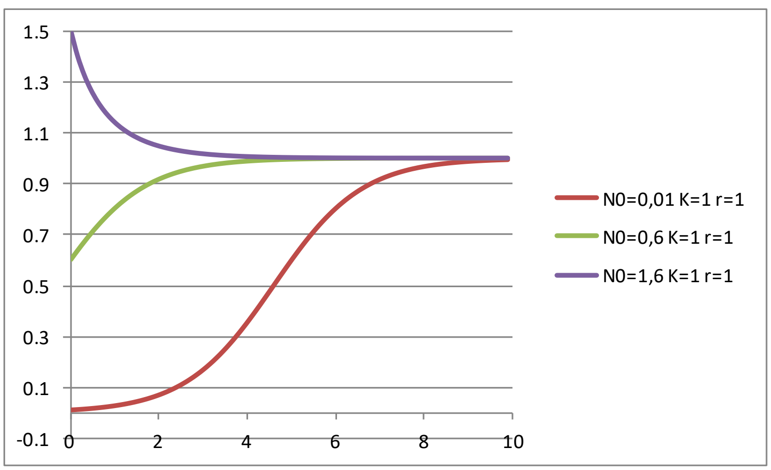
if $N < K$ population is growing

$$\frac{dN}{dt} = r(1 - \frac{N}{K})N$$

We assumed that $N(t) = N_0$

$$N(t) = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}}$$

$$\lim_{t\to\infty} \frac{KN_0}{N_0 + (K - N_0)e^{-rt}} = K$$



Modelowanie matematyczne w biologii i medycynie

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