MATRIX

A matrix is a rectangular array of numbers in which rows and columns can be distinguished. Matrices are usually marked with capital letters, e.g. A, B, C etc.

The rectangular matrix below has n rows and m columns, which is a dimension n x m.

$$A_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

We denote the element (matrix word) at the intersection of the i-th row and j-th column a_{ij}.

The element $a_{1,4}$ is in the first row in the fourth column.

The element $a_{3,2}$ is in the third row in the second column.

Hidden in the matrix are horizontal and vertical vectors.

Horizontal:

$$\vec{v}_1 = [a_{11} \quad a_{12} \quad \cdots \quad a_{1m}]$$

$$\vec{v}_2 = [a_{21} \quad a_{22} \quad \cdots \quad a_{2m}]$$

$$\vec{v}_n = [a_{n1} \quad a_{n2} \quad \cdots \quad a_{nm}]$$

Vertical:

$$\vec{u}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{n1} \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{n1} \end{bmatrix} \quad \vec{u}_n = \begin{bmatrix} a_{1m} \\ a_{2m} \\ \cdots \\ a_{nm} \end{bmatrix}$$

A matrix with the number of rows n and the number of columns m equal ($n = m$) is called a square matrix of degree n.

Example:

Enter the dimension of the following matrices

$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix}$ $\qquad A_{2\times 3}$

$$B = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix}$$ $B_{3\times3}$, that is, a square matrix of degree 3

$$C = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$ $C_{4\times1}$

$$D = \begin{bmatrix} 1 & -2 & 3 & 4 \end{bmatrix}$$ $D_{1\times4}$

$$E = [3]$$ $E_{1\times1}$

- Matrix addition and subtraction

The dimensions of the matrices to be added or subtracted must be the same

$$A_{n\times m}+B_{n\times m} =$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2m}+b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \dots & a_{nm}+b_{nm} \end{bmatrix}$$

$$A_{n\times m}-B_{n\times m} =$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \dots & a_{1m}-b_{1m} \\ a_{21}-b_{21} & a_{22}-b_{22} & \dots & a_{2m}-b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}-b_{n1} & a_{n2}-b_{n2} & \dots & a_{nm}-b_{nm} \end{bmatrix}$$

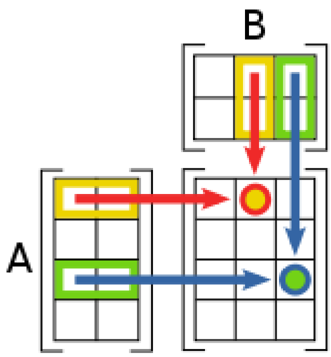
- Multiply a matrix by a number

$$k \cdot A_{n\times m} = k \cdot \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} = \begin{bmatrix} k\cdot a_{11} & k\cdot a_{12} & \dots & k\cdot a_{1m} \\ k\cdot a_{21} & k\cdot a_{22} & \dots & k\cdot a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ k\cdot a_{n1} & k\cdot a_{n2} & \dots & k\cdot a_{nm} \end{bmatrix}$$

- Matrix transposition

$$A_{n\times m}{}^{T} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}^{T} = C_{m\times n} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix}$$

- Multiplication of two matrices



https://pl.wikipedia.org/wiki/Mno%C5%BCenie_macierzy

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 7 & 5 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 38 & 36 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{bmatrix} 1 & 3 \\ 7 & 5 \\ 0 & 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & 1 \end{bmatrix} & \begin{bmatrix} 1\cdot 1 + 0\cdot 7 + 2\cdot 0 & 1\cdot 3 + 0\cdot 5 + 2\cdot 2 \\ 3\cdot 1 + 5\cdot 7 + 1\cdot 0 & 3\cdot 3 + 5\cdot 5 + 1\cdot 2 \end{bmatrix} \end{array}$$

- Identity matrix I

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$A_{m\times n} \cdot I_n = A_{m\times n} = I_m \cdot A_{m\times n}$$

- The order of operations on matrices:

Multiply before addition and subtraction

Transpose before other activities

First, the actions in parentheses

- Laws of actions on matrices:

$(A + B) + C = A + (B + C)$,

$(A \cdot B) \cdot C = A \cdot (B \cdot C)$,

$D + F = F + D$,

$k(D + F) = k D + k F$,

$k(DE) = (kD)E = D(kE)$,

$(A + B)E = AE + BE$,

$D(A + B) = DA + DB$,

$(D + F)^T = D^T + F^T$,

$(F^T)^T = F$,

$(FE)^T = E^T F^T$.

DETERMINANT of MATRIX

To each square matrix A we unambiguously assign a number of the so-called determinant of matrix A:

$$detA_{n\times n} = \det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

If the determinant of the matrix $\det A = 0$, then the matrix A is called singular.

The rules for calculating the determinant:

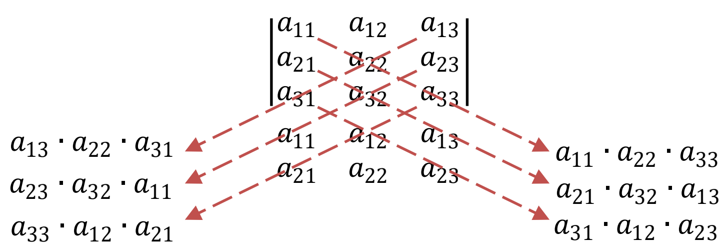
$$det[a] = |a| = a$$

$$det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

$$det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11} \cdot a_{22} \cdot a_{33} + a_{21} \cdot a_{32} \cdot a_{13} + a_{31} \cdot a_{12} \cdot a_{23} - (a_{13} \cdot a_{22} \cdot a_{31} + a_{23} \cdot a_{32} \cdot a_{11} + a_{33} \cdot a_{12} \cdot a_{21})$$

The rule of Sarrus:



$$a_{13} \cdot a_{22} \cdot a_{31} + a_{23} \cdot a_{32} \cdot a_{11} + a_{33} \cdot a_{12} \cdot a_{21}$$

$$a_{11} \cdot a_{22} \cdot a_{33} + a_{21} \cdot a_{32} \cdot a_{13} + a_{31} \cdot a_{12} \cdot a_{23}$$

Calculating the value of determinants of higher orders - Laplace's method

$$det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \sum_{j=1}^{n} a_{ij} det A_{ij}$$

where,

i - is fixed and specifies the row of the matrix against which to expand

a_{ij} - is a matrix element in the i-th row and in the j-th column

A_{ij} - is the algebraic complement of an element a_{ij}

Expand according to the elements of the second line:

$$det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = a_{21} \cdot (-1)^{2+1} \cdot det\begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} +$$

$$a_{22} \cdot (-1)^{2+2} \cdot det\begin{bmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{bmatrix} + a_{23} \cdot (-1)^{2+3} \cdot det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + \cdots$$

$$+a_{2n} \cdot (-1)^{2+n} \cdot det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} \\ a_{31} & a_{32} & \cdots & a_{3n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn-1} \end{bmatrix}$$

Expansion according to the elements of the third row:

$$det\begin{bmatrix} 5 & 3 & -1 & 2 \\ 2 & 0 & 4 & 3 \\ -3 & 6 & 2 & 0 \\ 4 & 0 & -5 & -2 \end{bmatrix} = (-3) \cdot (-1)^{3+1} \begin{vmatrix} 3 & -1 & 2 \\ 0 & 4 & 3 \\ 0 & -5 & -2 \end{vmatrix} + 6 \cdot (-1)^{3+2} \begin{vmatrix} 5 & -1 & 2 \\ 2 & 4 & 3 \\ 4 & -5 & -2 \end{vmatrix} +$$

$$2 \cdot (-1)^{3+3} \begin{vmatrix} 5 & 3 & 2 \\ 2 & 0 & 3 \\ 4 & 0 & -2 \end{vmatrix} + 0 \cdot (-1)^{3+4} \begin{vmatrix} 5 & 3 & -1 \\ 2 & 0 & 4 \\ 4 & 0 & -5 \end{vmatrix}$$

$$= (-3) \cdot 21 + 6 \cdot (-1) \cdot (-33) + 2 \cdot 48 + 0 \cdot (-1) \cdot 78 = 231$$

Expand according to the elements of the second column:

$$det\begin{bmatrix} 5 & 3 & -1 & 2 \\ 2 & 0 & 4 & 3 \\ -3 & 6 & 2 & 0 \\ 4 & 0 & -5 & -2 \end{bmatrix} = 3 \cdot (-1)^{1+2} \begin{vmatrix} 2 & 4 & 3 \\ -3 & 2 & 0 \\ 4 & -5 & -2 \end{vmatrix} + 0 \cdot (-1)^{2+2} \begin{vmatrix} 5 & -1 & 2 \\ -3 & 2 & 0 \\ 4 & -5 & -2 \end{vmatrix} +$$

$$6 \cdot (-1)^{3+2} \begin{vmatrix} 5 & -1 & 2 \\ 2 & 4 & 3 \\ 4 & -5 & -2 \end{vmatrix} + 0 \cdot (-1)^{4+2} \begin{vmatrix} 5 & -1 & 2 \\ 2 & 4 & 3 \\ -3 & 2 & 0 \end{vmatrix}$$

$$= 3 \cdot (-1) \cdot (-11) + 0 \cdot 0 + 6 \cdot (-1) \cdot (-33) + 0 \cdot 11 = 231$$

Other properties of determinants:

- the determinant of the transposed matrix is equal to the determinant of the output matrix,
- if the matrix has a zero row (zero column), then detA = 0,
- if the matrix has two identical rows (columns), then detA = 0,
- if a row (column) is a linear combination of other rows (columns), then detA = 0,
- swapping two rows or two columns of the matrix changes the sign of the determinant,
- if in a given matrix the elements of a given row or column are multiplied by any number $k \neq 0$, then the value of the determinant will also be multiplied by k,
- $\det(A \cdot B) = \det A \cdot \det B$ is the same
- the matrix determinant does not change the value if we add a linear combination of the remaining rows (or columns) to a row (or column) of the matrix

EQUATIONS:

Matrix notation of a system of linear equations:

System of n linear equations:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\dots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

It can be written in the matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

This corresponds to: $AX = B$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The matrix A is called the coefficient matrix of the system

Matrix X is a matrix that is a matrix of unknowns

Matrix B is an intercept matrix

CRAMER THEOREM

A system of linear equations is a Cramer's system when the number of equations is equal to the number of its unknowns and the main determinant (of the matrix of coefficients of the system) is not zero.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

If the determinant of the system's matrix of coefficients is not zero

(det A ≠ 0), then the system of linear equations has exactly one solution given by the formulas:

$$x_i = \frac{W_i}{W}$$

where: $W = detA$

W_i it is the determinant of the matrix that arises from the matrix A, by replacing the column of coefficients of the unknown x_i with the column of intercepts

$$W = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$W_i = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}$$

instead of the i-th column

The remaining determinants of W_i are created in the same way, replacing the appropriate column of coefficients in the matrix with a column of intercepts.

If the main determinant $W = 0$ and at least one determinant of W_i is different from zero, the system is inconsistent (there are no solutions).

If the main determinant $W = 0$ and all determinants $W_i = 0$ then the system is infinite (it has infinitely many solutions).

INVERSE MATRIX:

Inverse matrix A^{-1} of the square matrix A is called the matrix that satisfies the equality:

$$AA^{-1} = I, \qquad A^{-1}A = I$$

If the square matrix A is a non-singular matrix, i.e. det$A \neq 0$, then exactly one inverse matrix A^{-1} exists for it.

Example:

$$A = \begin{bmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{bmatrix}$$

$$detA = \begin{vmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{vmatrix} = -3$$

$$M = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 9 & 4 \\ 5 & 3 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 3 & 9 \\ 1 & 5 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 7 & 3 \\ 5 & 3 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 2 & 7 \\ 1 & 5 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 7 & 3 \\ 9 & 4 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 7 & -5 & 6 \\ -6 & 3 & -3 \\ 1 & 1 & -3 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 7 & -6 & 1 \\ -5 & 3 & 1 \\ 6 & -3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{detA}\begin{bmatrix} 7 & -6 & 1 \\ -5 & 3 & 1 \\ 6 & -3 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{3} & 2 & -\frac{1}{3} \\ \frac{5}{3} & -1 & -\frac{1}{3} \\ -2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} -\frac{7}{3} & 2 & -\frac{1}{3} \\ \frac{5}{3} & -1 & -\frac{1}{3} \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If the matrix A s non-singular, then by multiplying both sides by A^{-1} we get:

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

WolframAlpha

$$\begin{bmatrix}1 & 0\\3 & 5\end{bmatrix}+\begin{bmatrix}1 & 3\\7 & 5\end{bmatrix}$$

{{1,0},{3,5}}+{{1,3},{7,5}}

press „="

$$\begin{bmatrix}1 & 0 & 2\\3 & 5 & 1\end{bmatrix}^T$$

transpose {{1,0,2},{3,5,1}}

press „="

$$\begin{bmatrix}1 & 0 & 2\\3 & 5 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 3\\7 & 5\\0 & 2\end{bmatrix}=\begin{bmatrix}1 & 7\\38 & 36\end{bmatrix}$$

{{1,0,2},{3,5,1}}*{{1,3},{7,5},{0,2}}

press „="

$$\begin{bmatrix}1 & 0\\3 & 5\end{bmatrix}^3$$

{{1,0},{5,1}}^3

press „="

$$\begin{vmatrix}1 & 0\\3 & 5\end{vmatrix}$$

determinant{{1,0},{3,5}}

det{{1,0},{3,5}}

press „="

$$\begin{bmatrix}1 & 0\\3 & 5\end{bmatrix}^{-1}$$

inverse{{1,0},{3,5}}

press „="